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Classical Communication Cost and Remote Preparation of the Two-Atom Maximally Entangled State

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Abstract We present a protocol for probabilistic remote preparation of the two-atom maximally entangled state using a four-atom GHZ entangled state as the quantum channel. The two-atom maximally entangled state can be successfully prepared between two distant parties with a success probability of 100%. The successful total probability and classical communication cost are calculated.

Keywords Remote state preparation · Four-atom GHZ entangled state · Classical communication

Quantum entanglement and classical communication are two elementary resources in quantum information field. Quantum teleportation process, originally proposed by Bennett et al. [1], can transmit an unknown quantum state from a sender to a spatially distant receiver via a quantum channel with the help of some classical information. E.g., if Alice and Bob share an Einstein-Podolsky-Rosen (EPR) pairs, Alice can teleport a qubit to Bob by first carrying out a Bell-state measurement on the qubit and one particle of the EPR pair, then sending two bits of classical information to Bob, who in turn can perform a corresponding unitary operation on his particle (the other particle of the EPR pair) to get the state Alice wants to teleport. Recently, Lo [2] has presented an interesting new method to transmit pure known quantum state using a prior shared entanglement and some classical communication when the sender knows completely the transmitted state. This communication protocol is called remote state preparation (RSP). The main difference between RSP and teleportation are in that, (1) in RSP Alice knows the state that she wants Bob to prepare, in particular, Alice need not own the state, but only know the information about the state, while in teleportation

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Alice must own the teleported state, but she need not know the state; (2) in RSP, the required resource can be traded off between classical communication cost and entanglement cost while in quantum teleportation, two bits of forward classical communication and one ebit of entanglement (an EPR pair) per teleported qubit are both necessary and sufficient, and neither resource can be traded off against the other [4, 5].

RSP has attracted many attentions, in recent years [2–11]. Pati [3] has found RSP protocol more economical than teleportation and requires only one classical bit from Alice to Bob for some special ensembles. Bennett et al. [4] have generalized RSP for arbitrary qubits, higher dimensional Hilbert spaces and also of entangled systems. Ye et al. [5] have considered the faithful remote state preparation using finite classical bits and a non-maximally entangled state. Dai et al. have proposed a protocol [6] for probabilistic remote preparation of the four-particle GHZ class state using two partial entangled three-particle GHZ entangled states as quantum channel. Yu et al. have proposed a protocol [7, 8] for remote preparation of a qudit using maximally entangled states of qubits. In addition, some authors have also investigated the RSP protocols via using different quantum channel such as partial EPR pairs [9–11] and three-particle GHZ entangled state [12–14].

In this paper, we propose a protocol for deterministic remote preparation of a two-atom maximally entangled state using a four-atom GHZ entangled state as quantum channel. The total success probability and classical communication cost of successful RSP for the two-atom maximally entangled state are obtained. It is shown that the classical communication cost is greatly reduced, i.e., our protocol is more economical and consumes less classical communication than RSP one found by Pati [3] and may be known as the minimum classical communication protocol.

Suppose that Alice wants to help Bob remotely prepare a two-atom maximally entangled state

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + |ee\rangle), \tag{1}$$

one of the four EPR pairs. We suppose that Alice knows $|\phi\rangle$ completely, but Bob does not know it at all.

We suppose that the quantum channel shared by Alice and Bob is four-atom GHZ entangled state as follows

$$|\Psi\rangle_{1234} = \frac{1}{\sqrt{2}}(|gggg\rangle - i|eeee\rangle)_{1234}, \tag{2}$$

atoms (3, 4) belong to Alice while atoms (1, 2) belong to the receiver Bob. In order to help Bob to remotely prepare the two-atom maximally entangled state of (1), Alice must sent her atoms (3, 4) interact simultaneously with a single-mode cavity, at the same time the atoms are driven by a classical field (see Fig. 1). In the rotating-wave approximation, the Hamiltonian [15–17] for the system is

$$H = \omega_0 S_z + \omega_a a^+ a + \sum_{j=1}^{2}\left[\mathrm{g}\left(a^+ S_j^- + a S_j^+\right) + \Omega\left(S_j^+ e^{-i\omega t} + S_j^- e^{i\omega t}\right)\right], \tag{3}$$

were $S_z = \frac{1}{2}\sum_{j=1,2}(|e_j\rangle\langle e_j| - |g_j\rangle\langle g_j|)$, $S_j^+ = |e_j\rangle\langle g_j|$, $S_j^- = |g_j\rangle\langle e_j|$, and $|e_j\rangle$, $|g_j\rangle$ are the excited and ground states of the jth atom, respectively. a^+, a are the creation and annihilation operators for the cavity mode, and g is the atom-cavity coupling strength, Ω is the Rabi frequency, ω_0 is the atomic transition frequency, ω_a is the cavity frequency, and ω is the

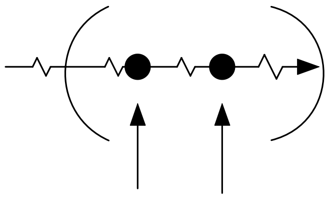

Fig. 1 Two atoms in a single-mode cavity and a classical field shown by a wavy line

frequency of the classical field. Assuming $\omega_0 = \omega$, in the interaction picture, the interaction Hamiltonian is

$$H_I = \Omega \sum_{j=1,2} (S_j^+ + S_j^-) + g \sum_{j=1}^{2} \big(e^{-i\delta t} a^+ S_j^- + e^{i\delta t} a S_j^+\big), \tag{4}$$

where δ is the detuning between the atomic transition frequency ω_0 and cavity frequency ω_a.

We assume in the strong driving regime $\Omega \gg \delta$, g and can neglect the terms oscillating fast. In the case $\delta \gg g$, there is no energy exchange between the atomic system and the cavity. Then in the interaction picture, the effective interaction Hamiltonian reads [18]

$$H_e = \frac{\chi}{2}\left[\sum_{h=1}^{2} (|e\rangle_{jj}\langle e| + |g\rangle_{jj}\langle g|) + \sum_{i,j=1,\ i\neq j}^{2} (S_i^+ S_j^- + S_i^+ S_j^+ + \text{H.c.})\right], \tag{5}$$

where $\chi = g^2/2\delta$. We note that the effective Hamiltonian is independent of the cavity field state, allowing it to be in a thermal state.

Then the evolution operator of the system is given by

$$U(t) = e^{-iH_0 t} e^{-iH_e t}, \tag{6}$$

where

$$H_0 = \Omega \sum_{j=1,\ 2} (S_j^+ + S_j^-). \tag{7}$$

After an interaction time τ Alice and Bob have

$$\begin{aligned} |\Psi\rangle_{1234} = \frac{e^{-i2\chi\tau}}{\sqrt{2}} & (|gg\rangle_{12}\{\cos(\chi\tau)[\cos(\Omega\tau)|g\rangle_3 \\ & - i\sin(\Omega\tau)|e\rangle_3][\cos(\Omega\tau)|g\rangle_4 - i\sin(\Omega\tau)|e\rangle_4] \\ & - i\sin(\Omega\tau)[\cos(\Omega\tau)|e\rangle_3 - i\sin(\Omega\tau)|g\rangle_3][\cos(\Omega\tau)|e\rangle_4 \\ & - i\sin\Omega\tau|g\rangle_4]\} - i|ee\rangle_{12}\{\cos(\chi\tau)[\cos(\Omega\tau)|e\rangle_3 \\ & - i\sin(\Omega\tau)|g\rangle_3][\cos(\Omega\tau)|e\rangle_4 - i\sin(\Omega\tau)|g\rangle_4] \\ & - i\sin(\chi\tau)[\cos(\Omega\tau)|g\rangle_3 - i\sin(\Omega\tau)|e\rangle_3] \\ & \times [\cos(\Omega\tau)|g\rangle_4 - i\sin(\Omega\tau)|e\rangle_4]\}). \end{aligned} \tag{8}$$

(*Case 1*) With the choice $\Omega\tau = \pi$, $\chi\tau = \pi/4$, the four-atom GHZ entangled state quantum channel is transformed into the maximally four-atom entangled state

$$|\Psi\rangle_{1234} = \frac{1}{2}(|gggg\rangle - i|ggee\rangle - |eegg\rangle - i|eeee\rangle), \tag{9}$$

where a common phase factor $e^{-i\pi/2}$ is discarded.

Using local operation, Alice and Bob can transform the state (9) into the entangled cluster state

$$|\Psi\rangle_{1234} = \frac{1}{2}(|gggg\rangle + |ggee\rangle + |eegg\rangle - |eeee\rangle). \tag{10}$$

Now, let us see how the original state described in (1) can be prepared form the state in (10). Firstly, Alice measures atoms (3, 4), respectively.

$$\begin{aligned}|\Psi\rangle_{1234} &= \frac{1}{2}(|gg\rangle_{34}(|gg\rangle + |ee\rangle)_{12} + |ee\rangle_{34}(|gg\rangle - |ee\rangle)_{12} \\ &= \frac{1}{\sqrt{2}}[|gg\rangle_{34}|\phi\rangle_{12} + |ee\rangle_{34}\frac{1}{\sqrt{2}}(|gg\rangle - |ee\rangle)_{12}].\end{aligned} \tag{11}$$

If Alice's von Neumann measurement result is $|gg\rangle_{34}$, atoms (1, 2) will collapse into

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + |ee\rangle)_{12}. \tag{12}$$

The remote preparation of a two-atom maximally entangled state is successfully realized and the probability that Alice measures result $|gg\rangle_{34}$ is 50%.

If Alice's measurement result is $|ee\rangle_{34}$, the state of atoms (1, 2) will collapse into the state

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|gg\rangle - |ee\rangle)_{12}. \tag{13}$$

Alice will inform Bob to preform the following unitary transformation U_1 (see Table 1) on atoms (1, 2).

$$U_1 = (|g\rangle\langle g| + |e\rangle\langle e|)_1 \otimes (|g\rangle\langle g| - |e\rangle\langle e|)_2. \tag{14}$$

That is, the unitary transformation U_1 in (14) will transform the state described in (13) into

$$|\phi\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + |ee\rangle)_{12}. \tag{15}$$

Then the remote preparation of two-atom maximally entangled state is successfully realized and the probability that Alice measures result $|ee\rangle_{34}$ is 50% too. Thus the total probability of successful RSP is $50\% + 50\% = 100\%$. The required amount of classical communication is 1 bit.

(*Case 2*) We consider the case of the non-maximally four-atom cluster state as quantum channel, (8) can rewrite into

$$|\Psi\rangle_{1234} = a|gggg\rangle + b|ggee\rangle + c|eegg\rangle - d|eeee\rangle, \tag{16}$$

Table 1 Corresponding relations among the unitary U_1 operation, the RSP states, and Alice's measurement results. The RSP states and Alice's measurement results are listed in the first column (line)

$\|\phi\rangle$	$\|gg\rangle_{34}$	$\|ee\rangle_{34}$
$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)_{12}$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_2$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|g\rangle\langle g\|-\|e\rangle\langle e\|)_2$
$\frac{1}{\sqrt{2}}(\|00\rangle-\|11\rangle)_{12}$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|g\rangle\langle g\|-\|e\rangle\langle e\|)_2$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_2$
$\frac{1}{\sqrt{2}}(\|01\rangle+\|10\rangle)_{12}$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|g\rangle\langle e\|+\|e\rangle\langle g\|)_2$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|e\rangle\langle g\|-\|g\rangle\langle e\|)_2$
$\frac{1}{\sqrt{2}}(\|01\rangle-\|10\rangle)_{12}$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|e\rangle\langle g\|-\|g\rangle\langle e\|)_2$	$(\|g\rangle\langle g\|+\|e\rangle\langle e\|)_1\otimes(\|e\rangle\langle g\|+\|g\rangle\langle e\|)_2$

where the coefficients a, b, c, and d satisfy $|a|^2+|b|^2+|c|^2+|d|^2=1$, and $|a|\geq|b|\geq|c|\geq|d|$. In order to help Bob to remotely prepare the two-atom maximally entangled state of (1), Alice must perform a projective measurement on atoms (3, 4), respectively.

$$|\Psi\rangle_{1234}=|gg\rangle_{34}(a|gg\rangle+c|ee\rangle)_{12}+|ee\rangle_{34}(b|gg\rangle-d|ee\rangle)_{12}. \tag{17}$$

If Alice's von Neumann measurement result is $|gg\rangle_{34}$, the probability that Alice measures result $|gg\rangle_{34}$ is $\frac{a^2+c^2}{a^2+b^2+c^2+d^2}$, then the state of atoms (1, 2) will collapse into

$$(a|gg\rangle+c|ee\rangle)_{12}. \tag{18}$$

After normalization, the state in (18) will be

$$\frac{1}{\sqrt{a^2+c^2}}(a|gg\rangle+c|ee\rangle)_{12}. \tag{19}$$

Now, let us see how the original state described in (1) can be prepared from the state in (19). Firstly Alice will inform Bob of her measurement outcome via a classical communication. According to Alice's measurement result, Bob introduces an auxiliary two-level atom A with the initial state $|g\rangle_A$, the state described in (19) will become

$$\frac{1}{\sqrt{a^2+c^2}}(a|gg\rangle+c|ee\rangle)_{12}\otimes|g\rangle_A. \tag{20}$$

Bob performs another unitary transformation U_2 on atoms 1 and A under the basis $\{|gg\rangle_{1A},|eg\rangle_{1A},|ge\rangle_{1A},|ee\rangle_{1A}\}$, the unitary transformation U_2 may take the form of the following 4×4 matrix

$$U^B=\begin{pmatrix} \frac{c}{a} & 0 & \sqrt{1-\frac{c^2}{a^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\frac{c^2}{a^2}} & 0 & -\frac{c}{a} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{21}$$

The unitary transformation U_2 will transform the state in (20) into

$$\frac{\sqrt{2}c}{\sqrt{a^2+c^2}}|\phi\rangle_{12}\otimes|g\rangle_A+\frac{\sqrt{a^2-c^2}}{\sqrt{a^2+c^2}}|gg\rangle_{12}\otimes|e\rangle_A. \tag{22}$$

Finally, Bob measures the state of auxiliary atom A. If his measurement result is $|e\rangle_A$, the remote preparation of the original state fails. If the result $|g\rangle_A$ is measured, the remote preparation of two-atom maximally entangled state is successfully realized and the probability that Bob measures result $|g\rangle_A$ is $\frac{2c^2}{a^2+c^2}$, thus the probability of successful RSP is $\frac{a^2+c^2}{a^2+b^2+c^2+d^2} \times \frac{2c^2}{a^2+c^2} = \frac{2c^2}{a^2+b^2+c^2+d^2}$. The required amount of classical communication is $2c^2 \log_2[\frac{1}{2c^2}]$ bits.

Similarly, from (17), if Alice's measurement result on atoms (3, 4) is $|ee\rangle_{34}$, the probability that Alice measures result $|ee\rangle_{34}$ is $\frac{b^2+d^2}{a^2+b^2+c^2+d^2}$, the state of atoms (1, 2) will become

$$(b|gg\rangle - d|ee\rangle)_{12}. \tag{23}$$

After normalization, the state in (23) will become

$$\frac{1}{\sqrt{b^2+d^2}}(b|gg\rangle - d|ee\rangle)_{12}. \tag{24}$$

Now Bob operates the following unitary transformation on the state of (24)

$$U_1 = (|g\rangle\langle g| + |e\rangle\langle e|)_1 \otimes (|g\rangle\langle g| - |e\rangle\langle e|)_2, \tag{25}$$

which can transform the state shown in (24) into

$$\frac{1}{\sqrt{b^2+d^2}}(b|gg\rangle + d|ee\rangle)_{12}. \tag{26}$$

Bob introduces an auxiliary atom A with the initial state $|g\rangle_A$ and performs an unitary transformation U_3 on atom 1 and A.

$$U^B = \begin{pmatrix} \frac{d}{b} & 0 & \sqrt{1-\frac{d^2}{b^2}} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\frac{d^2}{b^2}} & 0 & -\frac{d}{b} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{27}$$

The unitary transformation U_3 will transform the state in (26) into

$$\frac{\sqrt{2}d}{\sqrt{b^2+d^2}}|\phi\rangle_{12} \otimes |g\rangle_A + \frac{\sqrt{b^2-d^2}}{\sqrt{b^2+d^2}}|gg\rangle_{12} \otimes |e\rangle_A. \tag{28}$$

Obviously, Bob can construct the two-atom maximally entangled state that Alice wishes to prepare remotely and the probability that Bob measures result $|g\rangle_A$ is $\frac{2d}{b^2+d^2}$. So the probability of successful RSP is $\frac{b^2+d^2}{a^2+b^2+c^2+d^2} \times \frac{2d^2}{b^2+d^2} = \frac{2d^2}{a^2+b^2+c^2+d^2}$. Thus the total probability of successful RSP using a non-maximally entangled cluster state is $\frac{2c^2}{a^2+b^2+c^2+d^2} + \frac{2d^2}{a^2+b^2+c^2+d^2} = \frac{2c^2+2d^2}{a^2+b^2+c^2+d^2}$. It also consumes $2d^2 \log_2[\frac{1}{2d^2}]$ bits of classical communication.

From the above analysis, we can see that, in Case 1, the total probability of successful RSP is 100%; in Case 2, the total probability of successful RSP is $\frac{2c^2+2d^2}{a^2+b^2+c^2+d^2}$. If $|a|^2 = |b|^2 = |c|^2 = |d|^2 = \frac{1}{2}$, the total probability of successful RSP equals 1, and the total required amount of classical communication is 1 ($2c^2 \log_2[\frac{1}{2c^2}] + 2d^2 \log_2[\frac{1}{2d^2}]$) bit. Next we give a brief discussion on the experimental matters. For the Rydberg atoms, the radiative time is about $T_r = 3 \times 10^{-2}$ s, and the coupling constant is $g = 2\pi \times 24$ kHz [19]. The required atom-cavity-field interaction time is on the order of $T \approx 10^{-4}$ s. Then the time needed to complete the whole procedure is much shorter than T_r. Meanwhile it is noted that the atomic state evolution is independent of the cavity field state, thus the cavity decay will not affect the generation of the four-atom entangled cluster state.

In summary, we present a protocol for probabilistic RSP of the two-atom maximally entangled state using a four-atom GHZ entangled state with cavity QED. In this paper, two different cases (Case 1, and Case 2) for RSP are proposed. The two-atom maximally entangled state can be perfectly prepared if the sender Alice sends his atoms (3, 4) into a cavity QED using the interaction of two two-level atoms with a single-mode nonresonant cavity with the assistance of a strong classical driving field, and she performs a von Neumann measurement on atoms (3, 4), respectively, and the receiver introduces an appropriate unitary transformation with the help of entanglement and classical communication. We also considered the case of four-atom non-maximally entangled cluster state as quantum channel. The result show that, for such a four-atom non-maximally entangled cluster state quantum channel, there is still a certain probability of successful RSP. Compared with the previous RSP protocol [3], the present one has the following advantages. Firstly, the quantum channel is different in forms. Secondly, we have calculated the successful total probability and total classical communication of probabilistic RSP protocol, which have relation to the two smaller coefficients of non-maximally state used as quantum channel (in Case 2). Thirdly, in our protocol, it is just need single atom measurement. Fourthly, our results indicate that such remote preparation of the two-atom maximally entangled state requires only 1 bit classical communication cost for the maximally entangled channel, namely, our protocol is more economical and consumes less classical communication than RSP one found by Pati [3], classical communication cost is greatly reduced. Thus from the point of view of communication theory, our protocol is optimal and uses the minimum classical bits. It may be useful not only in understanding the essence of the classical communication in RSP process, but also expanding the applied field of classical information science. Nowadays, four-particle entanglement has also been demonstrated in ion traps [20], a number of feasible protocols for generating GHZ entangled states have been proposed [21–23], therefore we believe that this protocol may be realized in the realm of current experimental technology.

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